

Global existence of small amplitude solutions to nonlinear coupled wave-Klein-Gordon systems in four space-time dimension with hyperboloidal foliation method

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Abstract

In this article one will develop a new type of energy method based on a foliation of space-time into hyperboloidal hypersurfaces. As we will see, with this method, some classical results such as global existence and almost global existence of regular solutions to the quasi-linear wave equations and Klein-Gordon equations will be established in a much simpler and much more natural way. Most importantly, the global existence of regular solutions to a general type of coupled quasilinear wave-Klein-Gordon system will be established. All of this suggests that compared with the classical method, this hyperboloidal foliation of space-time may be a more natural way to regard the wave operator.

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1 Introduction

In the research of quasilinear hyperbolic partial differential equations, the global existence of regular solutions is a central problem. There are already lots of excellent works. S. Klainerman has firstly developed the conformal Killing vector fields method. With this method, he has managed to establish the global existence of regular solution to quasilinear wave equations with classical null conditions in \mathbb{R}^{3+1} (see [3]), and latter, global existence of regular solution to quasilinear Klein-Gordon equations in \mathbb{R}^{3+1} (see [4]). From that time, this conformal Killing vector fields method has been developed and applied by many other to many more general cases.

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However, because of an essential difference between wave equation and Klein-Gordon equation, one of the conformal Killing vector field of wave equation, the scaling field S , is not a conformal Killing vector field of Klein-Gordon equation. More unfortunately, this scaling vector field plays an important role in the decay estimates of wave equations. This leads to an essential difficulty when one attempt to establish the global existence of coupled wave-Klein-Gordon system. However in [2], S. Katayama has established in a relatively special case the global in time existence of this kind of system with a technical $L^\infty - L^\infty$ type estimate.

In another hand, L. Hörmander has developed an “alternative energy method” (see [1]) for dealing the global existence of quasilinear Klein-Gordon equation. His observation is as follows. Consider the following Cauchy problem associated to the linear Klein-Gordon equation in \mathbb{R}^{n+1}

$$(1.1) \quad \begin{cases} \square u + a^2 u = f, \\ u(B+1, x) = u_0, \quad u_t(B+1, x) = u_1, \end{cases}$$

where u_0, u_1 are regular functions supported on $\{(B+1, x) : |x| \leq B\}$ and f is also a regular function supported on

$$\Lambda' := \{(t, x) : |x| \leq t-1\},$$

with $a, B > 0$ two fixed positive constants. By the Huygens’ principle, the regular solution of (1.1) is supported in

$$\Lambda' \cap \{t \geq B+1\}.$$

One denotes by:

$$H_T := \{(t, x) : t^2 - x^2 = T^2, t > 0\}$$

and

$$G_{B+1} = \Lambda' \cap \{(t, x) : \sqrt{t^2 - x^2} \geq B+1\},$$

one can develop a hyperboloidal foliation of G_{B+1} , which is

$$G_{2B} = H_T \times [B+1, \infty).$$

Then, taking $\partial_t u$ as multiplier, the standard procedure of energy estimate leads one to the following energy inequality

$$E_m(H_T, u)^{1/2} \leq E_m(H_{B+1}, u)^{1/2} + \int_{B+1}^T ds \left(\int_{H_s} f^2 \right)^{1/2},$$

where

$$E_m(H_s, u) := \int_{H_s} \sum_{i=1}^3 ((x^i/t)\partial_t u + \partial_i u)^2 + ((T/t)\partial_t u)^2 + (a/2)u^2 dx.$$

Then, Hörmander has developed a Sobolev type estimate, see the lemma 7.6.1 of [1]. Combined with the energy estimate, he has managed to establish the decay estimate:

$$\sup_{H_T} t^{n/2} |u| \leq \sum_{|I| \leq m_0} E_m(H_T, Z^I u)^{1/2} \leq E_m(H_{B+1}, Z^I u)^{1/2} + \int_{B+1}^T ds \left(\int_{H_s} Z^I f^2 \right)^{1/2},$$

where m_0 is the smallest integer bigger the $n/2$.

But in the proof of [1], the only used term of the energy $E_m(H_T, u)$ is the last term u^2 . The first two terms seem to be omitted, at least when doing decay estimates. The new observation in this article is that the first two terms of the energy can also be used for estimating some important derivatives of the solution. This leads one to the possibility of applying this method on the case where $a = 0$, which is the wave equation, so that the wave equations and the Klein-Gordon equations can be treated in the same framework. That is the key of dealing the coupled wave-Klein-Gordon system, and one may call it hyperboloidal foliation method.

The following is a prototype of the main result which will be established in this article. Consider the Cauchy problem associated to the coupled wave-Klein-Gordon system in \mathbb{R}^{3+1} :

$$(1.2) \quad \begin{cases} \square u = N(\partial u, \partial u) + Q_1(\partial v, \partial v) + Q_2(\partial u, \partial v), \\ \square v + v = Q_3(\partial u, \partial u) + Q_4(\partial v, \partial v) + Q_5(\partial u, \partial v), \\ u(B+1, x) = \varepsilon u_0, \quad u_t(B+1, x) = \varepsilon u_1, \\ v(B+1, x) = \varepsilon v_0, \quad v_t(B+1, x) = \varepsilon v_1. \end{cases}$$

Here the $N(\cdot, \cdot)$ are the classical null quadratic terms while $Q_i(\cdot, \cdot)$ are arbitrary quadratic terms. u_0, u_1 are regular functions supported on $\{(B+1, x) : |x| \leq B\}$. As we will see in the following, in general the global existence result holds

Theorem 1.1 (Prototype of the main result). *There exists a $\varepsilon_0 > 0$ such that for any $0 \leq \varepsilon \leq \varepsilon_0$, (1.2) has an unique global in time regular solution.*

This result has already been established in [2] by S. Katayama. However, the advantages of the hyperboloidal foliation method are as follows. First, it provides a proof much simpler than that of [2] such that compared with the technical $L^\infty - L^\infty$ estimates, it will use nothing else but the energy estimates and the Sobolev type inequalities in lemma 7.6.1 of [1]. Second, when one add terms such as $\partial_\alpha \partial_{tt} w_j$ to the system, the method of [2] does not work any more while the hyperbolic method still works.

Furthermore, this new method provides much simpler proofs when applied to many classical problems such as the global existence of quasilinear wave equation in \mathbb{R}^{3+1} with null conditions.

The structure of this article is as follows. In section 2, one will introduce the notation and establish the basic estimates. In section 3, one will establish the main result of this article, the global in time existence of regular solution to coupled quasilinear wave-Klein-Gordon system.

2 Notation and Basic estimates

2.1 Notation and general framework

One denotes by H_T the hyperboloid $\{t^2 - |x|^2 = T^2\}$ with radius $T > 0$. Let

$$\Lambda' = \{|x| \leq t-1\},$$

and

$$G_{T_1}^{T_2} = \{|x| \leq t-1, T_1 \leq \sqrt{t^2 - |x|^2} \leq T_2\}.$$

On $H_T \cap \Lambda'$, when $T \geq 1$ one has

$$T \leq t \leq T^2,$$

while on $H_T \cap \{r := |x| \leq t/2\}$,

$$T \leq t \leq \sqrt{2}T.$$

Define the vector fields

$$H_i = t\partial_i + x^i\partial_t.$$

One sets $Z_\alpha := \partial_\alpha$ for $\alpha = 0, \dots, 3$, and $Z_\alpha := H_{\alpha-4}$ for $\alpha = 5, 6, 7$. One denotes also by Z^J the $|J|$ -th order operator $Z_{J_1} \cdots Z_{J_{|J|}}$, where J is a multi-index with length $|J|$. One notices that H_j are tangent to H_T and $\bar{\partial}_i := t^{-1}H_i$ is the projection of ∂_i on H_T . Moreover if one uses $\{x^i\}$ as a coordinate system on H_T , then $\bar{\partial}_i$ can also be regarded as the natural frame associated with these coordinates. on the tangent bundle on H_T , and the associated area element of H_T is

$$d\sigma = t^{-1}\sqrt{t^2 + |x|^2} dx.$$

One also defines the tangential derivatives $\bar{\partial}_i$,

$$\bar{\partial}_i = \omega^i \partial_t + \partial_i,$$

where $\omega^i := x^i/r$. Theses vector fields are tangent to the out-going light cone.

2.2 Basic energy estimates

One considers the following linear wave or Klein-Gordon equation:

$$(2.1) \quad \begin{cases} \square u + a^2 u = f, \\ u|_{H_{B+1}} = u_0, \quad \partial_t u|_{H_{B+1}} = u_1, \end{cases}$$

where a is a nonnegative constant and u_1, u_0 are regular functions supported on $H_{B+1} \cap \Lambda'$. f is also supposed to be a regular function with its support contained in $\Lambda' \cap G_{B+1}^\infty$. Define the energy on the hyperboloid H_T :

$$(2.2) \quad \begin{aligned} E_m(T, u) &:= \int_{H_T} \left(|\partial_t u|^2 + \sum_{i=1}^3 |\partial_i u|^2 + \frac{2x^i}{t} \partial_t u \partial_i u + 2(au)^2 \right) dx, \\ &= \int_{H_T} 2(au)^2 + \sum_{i=1}^3 |\bar{\partial}_i u|^2 + \left(\frac{T}{t} \partial_t u \right)^2 dx, \\ &= \int_{H_T} 2(au)^2 + \sum_{i=1}^3 \left(\frac{T}{t} \partial_i u \right)^2 + \sum_{i=1}^3 \left(\frac{r}{t} \partial_i u + \frac{x^i}{r} \partial_t u \right)^2 dx. \end{aligned}$$

Proof. See section 7.7 of [1]. □

Note that

$$|(\partial_i - \bar{\partial}_i)u| = \left| \frac{T\omega^i}{2t} \right| \left| \frac{T}{t} \partial_t u \right|,$$

which implies

$$\int_{H_T} \sum_{i=1}^3 |\partial_i u|^2 dx \leq E(T, u).$$

In general one has the following energy estimate:

Lemma 2.1 (Energy estimates). *Let u be a regular solution of (2.1) then the following energy estimate holds:*

$$(2.3) \quad E_m(T, u) \leq E_m(B+1, u) + \int_{B+1}^T \left(\int_{H_s} f^2 dx \right)^{1/2} ds.$$

In Appendix A one will see that for the case treated in this article, the energy defined on hyperboloid is controlled by the standard energy. More precisely, for any $\varepsilon, C_1 > 0$, there exists an $\varepsilon' = \varepsilon'(B, \varepsilon) > 0$ such that if

$$E^*(B+1, u)^{1/2} \leq \varepsilon',$$

then

$$E_m(B+1, u) \leq \varepsilon.$$

2.3 Commutators

In this subsection one discusses some commutative relations. They are very important for establishing the decay estimates. Recall that $T^2 = t^2 - r^2$. The following commutative properties are obvious:

$$(2.4) \quad [H_j, \square] = 0,$$

and

$$(2.5) \quad [\partial_\alpha, \square] = 0.$$

One defines the vector field family

$$\mathcal{D}_g := \{\partial_i, (t-r)r^{-1}\partial_\alpha\}.$$

Firstly one has the following commutative relations between H_j and ∂_α :

$$(2.6) \quad \begin{aligned} H_j \partial_t u &= \partial_t H_j u - \partial_j u, \\ H_j \partial_i u &= \partial_i H_j u - \delta_j^i \partial_t u. \end{aligned}$$

Notice that one has

$$H_j \left(\frac{T}{t} \right) = -\frac{x^j T}{t^2},$$

so one gets the following commutative relations between H_j and $\frac{T}{t}\partial_\alpha$:

$$(2.7) \quad \begin{aligned} H_j \left(\frac{T}{t} \partial_t u \right) &= -\frac{T}{t} \left(\partial_j u + \frac{x^j}{t} \partial_t u \right) + \frac{T}{t} \partial_t (H_j u), \\ H_j \left(\frac{T}{t} \partial_i u \right) &= -\frac{T}{t} \left(\delta_j^i \partial_t u + \frac{x^j}{t} \partial_i u \right) + \frac{T}{t} \partial_t (H_j u). \end{aligned}$$

The commutative relations between H_j and $D_g \in \mathcal{D}_g$ are

$$(2.8) \quad \begin{aligned} H_j \partial_i u &= \partial_i H_j u - \omega^i \partial_j u + (\delta_j^i - \omega^i \omega^j) \left(\frac{t}{r} - 1 \right) \partial_t, \\ H_j \left(\left(\frac{t}{r} - 1 \right) \partial_t u \right) &= \left(\frac{t}{r} - 1 \right) \partial_t (H_j u) - \omega^j \left(\frac{t}{r} + 1 \right) \left(\frac{t}{r} - 1 \right) \partial_t u - \left(\frac{t}{r} - 1 \right) \partial_j u, \\ H_j \left(\left(\frac{t}{r} - 1 \right) \partial_i u \right) &= \left(\frac{t}{r} - 1 \right) \partial_i (H_j u) - \omega^j \left(\frac{t}{r} + 1 \right) \left(\frac{t}{r} - 1 \right) \partial_i u - \delta_j^i \left(\frac{t}{r} - 1 \right) \partial_t u. \end{aligned}$$

The commutative properties between H_j and $\bar{\partial}_i$ are

$$(2.9) \quad H_j \bar{\partial}_i u = \bar{\partial}_j H_i u - \frac{x^j}{t} \bar{\partial}_j u.$$

In general one has the following results:

Lemma 2.2. *Let I be an arbitrary multi-index. In the region $\Lambda' \cap G_{B+1}^\infty$ one has:*

$$(2.10) \quad |H^I \partial_\alpha u| \leq |\partial_\alpha H^I u| + C(n, |I|) \sum_{|J| < |I|} \sum_{\beta=0}^n |\partial_\beta H^J u|,$$

$$(2.11) \quad \sum_{i=1}^3 \sum_{|I|=p} |H^I \bar{\partial}_i u| \leq \sum_{i=1}^3 \sum_{|I|=p} |\bar{\partial}_i H^I u| + C(n, |I|) \sum_{|J| < p} \sum_{j=1}^n |\bar{\partial}_j H^J u|,$$

and

$$(2.12) \quad \left| H^I \left(\frac{T}{t} \partial_\alpha u \right) \right| \leq \left| \frac{T}{t} \partial_\alpha H^I u \right| + C(n, |I|) \sum_{|J| < |I|} \sum_{\beta=0}^3 \left| \frac{T}{t} \partial_\beta H^J u \right|.$$

In the region $\{r \geq t/2\} \cap \Lambda' \cap G_{B+1}^\infty$ one has:

$$(2.13) \quad |H^I D_g u| \leq |D_g H^I u| + C(n, |I|) \sum_{\substack{|J| < |I| \\ Y \in \mathcal{D}_g}} |Y H^J u|,$$

where $C(n, |I|)$ is a constant depending only on dimension n and $|I|$.

Proof. The inequality (2.10) is a direct result of (2.6). To prove the (2.13), one notices that in (2.8), one has some non-constant coefficients. To get the result, one calculates the their derivatives,

$$\begin{aligned} H_j(\omega^i) &= (\delta_j^i - \omega^i \omega^j) \frac{t}{r}, \\ H_j\left(\frac{t}{r}\right) &= \frac{x^j}{r} - \frac{t^2 x^j}{r^3} = -\omega^j \left(\frac{t}{r} + 1\right) \left(\frac{t}{r} - 1\right), \\ H_j\left(\frac{t}{r} + 1\right) &= \frac{x^j}{r} - \frac{t^2 x^j}{r^3} = -\omega^j \left(\frac{t}{r} - 1\right) \left(\frac{t}{r} + 1\right), \\ H_j\left(\frac{t}{r} - 1\right) &= \frac{x^j}{r} - \frac{t^2 x^j}{r^3} = -\omega^j \left(\frac{t}{r} + 1\right) \left(\frac{t}{r} - 1\right), \end{aligned}$$

One denotes by \mathcal{F} the family of functions:

$$\mathcal{F} := \left\{ \omega, \frac{t}{r}, \frac{t}{r} + 1, \frac{t}{r} - 1 \right\}.$$

Then one gets, for arbitrary multi-index,

$$H^I D_g u = D_g H^I u + \sum_{|J| < |I|} F^{K(I,J)} D_g H^J u,$$

where $K(I, J)$ is a multi-index of length $|K|$ depending only on two other multi-index I and J , and F^K is a multiplier defined as follows:

$$F^K := F^{K_1} F^{K_2} \dots F^{K_{|K|}}, \quad F^{K_i} \in \mathcal{F}.$$

Notice that when $t/2 \leq r \leq t-1$, all the terms on the right-hand-side are bounded, which gives (2.13).

The inequalities (2.11) and (2.12) are proved similarly, one omits the details. \square

One has also the following commutative relations between ∂_α and D_g :

$$\begin{aligned} \partial_j \partial_i &= (\delta_j^i r^{-1} - \omega^i \omega^j r^{-1}) \partial_t + \partial_i \partial_j, \\ \partial_t \partial_i &= \partial_i \partial_t, \\ \partial_j \left(\frac{t}{r} - 1 \right) \partial_t &= -\omega^j \frac{t}{r^2} \partial_t + \left(\frac{t}{r} - 1 \right) \partial_t \partial_j, \\ \partial_t \left(\frac{t}{r} - 1 \right) \partial_t &= \frac{1}{r} \partial_t + \left(\frac{t}{r} - 1 \right) \partial_t \partial_t, \\ \partial_j \left(\frac{t}{r} - 1 \right) \partial_i &= -\omega^j \frac{t}{r^2} \partial_i + \left(\frac{t}{r} - 1 \right) \partial_i \partial_j, \\ \partial_t \left(\frac{t}{r} - 1 \right) \partial_i &= \frac{1}{r} \partial_i + \left(\frac{t}{r} - 1 \right) \partial_i \partial_t. \end{aligned} \tag{2.14}$$

And, the following result:

Lemma 2.3. *Let u be a regular function on $\Lambda' \cap G_{B+1}^\infty$. The the following estimates are true*

$$\begin{aligned} \sum_{|J|=p} |Z^J \bar{\partial}_i u| &\leq C(p, n) \sum_{|\beta| \leq p} \sum_{j=1}^3 |\bar{\partial}_j Z^\beta u|, \\ |Z^I \partial_\alpha u| &\leq |\partial_\alpha Z^I u| + C(|I|, n) \sum_{|J| < |I|} \sum_{\alpha=0}^3 |\partial_\alpha Z^J u|. \end{aligned}$$

When $r \geq \frac{1}{2}t$, then the following estimate is true

$$|Z^I \partial_i u| \leq |\partial_i Z^I u| + C(|I|, n) \sum_{|J| < |I|} \sum_{j=1}^3 |\partial_j Z^J u| + C(|I|, n)(T^2/t^2) \sum_{|J| < |I|} \sum_{\alpha=0}^3 |\partial_\alpha Z^I u|.$$

Proof. Considering the commutative relation (2.14), the proof is the same to that of lemma 2.2. \square

2.4 Frame and the null conditions

In this subsection, a so called “one-one” frame, denoted by $\{\underline{\partial}_\alpha\}$, will be introduced. Here

$$\begin{aligned} \underline{\partial}_0 &:= \partial_t, \\ \underline{\partial}_i &:= \partial_i, \quad i = 1, 2, 3. \end{aligned}$$

The transition matrix between this frame and the natural frame is

$$\Phi := \begin{pmatrix} 1 & 0 & 0 & 0 \\ \omega^1 & 1 & 0 & 0 \\ \omega^2 & 0 & 1 & 0 \\ \omega^3 & 0 & 0 & 1 \end{pmatrix},$$

so that

$$\underline{\partial}_\alpha u = \Phi_\alpha^\beta \partial_\beta u.$$

Its inverse is

$$\Psi := \Phi^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\omega^1 & 1 & 0 & 0 \\ -\omega^2 & 0 & 1 & 0 \\ -\omega^3 & 0 & 0 & 1 \end{pmatrix},$$

so that

$$\partial_\alpha u = \Psi_\alpha^\beta \underline{\partial}_\beta u.$$

The advantage of this “one-one” frame is that the last three vector fields are tangent to the outgoing light cone. In these directions, the gradient of the solution has better decay near the light cone. For a general two tensor \mathcal{T} , one can write it in the nature frame as

$$\mathcal{T} = T^{\alpha\beta} \partial_\alpha \partial_\beta,$$

or in the “one-one” frame

$$\mathcal{T} = \underline{T}^{\alpha\beta} \underline{\partial}_\alpha \underline{\partial}_\beta,$$

In general the following result holds:

Lemma 2.4. *For any two tensor \mathcal{T} one has, in the region $\Lambda' \cap \{r \geq t/2\}$,*

$$\sum_{\alpha, \beta=0}^3 |Z^I \underline{T}^{\alpha\beta}| \leq C(|I|, n) \sum_{|J| \leq |I|} \sum_{\alpha', \beta'=0}^3 |Z^J T^{\alpha'\beta'}|.$$

Proof. Note that in the region $\{r \geq t/2\} \cap \Lambda'$,

$$Z\omega \leq 1.$$

Then the proof is just a simple calculation. \square

Now it is the time to introduce the classical null conditions. Let $\{f_i\}$ be a finite set of regular function on \mathbb{R}^{n+1} . The following quadratic forms

$$A_i^{\alpha\beta\gamma j} \partial_\gamma f_j \partial_{\alpha\beta} f_i, \quad B_i^{\alpha\beta j} f_j \partial_{\alpha\beta} f_i, \quad P^{\alpha\beta ij} \partial_\alpha f_i \partial_\beta f_j$$

are said to satisfy the null conditions if for any $\xi \in \mathbb{R}^{n+1}$ such that $\xi_0 \xi_0 - \sum_{i=1}^3 \xi_i \xi_i = 0$,

$$(2.15) \quad A_i^{\alpha\beta\gamma j} \xi_\alpha \xi_\beta \xi_\gamma = B_i^{\alpha\beta j} \xi_\alpha \xi_\beta = P^{\alpha\beta ij} \xi_\alpha \xi_\beta = 0.$$

Clearly the following conditions are weaker than these null conditions:

$$(2.16) \quad \underline{A}_i^{000j} = \underline{B}_i^{00j} = \underline{P}^{00ij} = 0.$$

2.5 Basic decay estimates

For the convenience of statement, one defines the following norm:

$$\|u\|_{H,p,H_T}^2 := \sum_{|I| \leq p} \int_{H_T} |H^I u|^2 dx.$$

To turn L^2 estimates into L^∞ estimates, one needs the following Sobolev inequality, which is a slightly improvement of a result by Hörmander (see lemma 7.6.1 of [1]).

Lemma 2.5 (Sobolev-type estimate on hyperboloid). *Let $p(n)$ be the smallest integer $> n/2$. Any C^∞ function defined on \mathbb{R}^{1+n} satisfies*

$$(2.17) \quad \sup_{H_T} t^n |u(t, x)|^2 \leq C(n) \|u\|_{H,p(n),H_T}^2,$$

where $C(n) > 0$ is a constant depending only on dimension n .

Proof. One observes that the derivatives ∂_α are not actually used. □

Now it is the time to establish the basic decay estimates.

Lemma 2.6. *Suppose u is the regular solution of (2.1), then u satisfies the following decay estimates:*

$$(2.18) \quad \sup_{H_T} t^n (|\bar{\partial}_i u|^2 + |(T/t) \partial_\alpha u|^2 + |au|^2) + \sup_{H_T \cap \{r \geq t/2\}} t^n |\partial_i u|^2 \leq \sum_{|I| \leq p(n)} C(n) E_m(T, H^I u)$$

Proof. The proof is just a combination of energy estimat(2.3), the commutation estimate (2.2) and the Sobolev inequality (2.5). □

Remark 2.7. *From lemma 2.6, one easily gets the following result. Taking $f = 0$ and $g^{\alpha\beta} = m^{\alpha\beta}$, then the solution of homogeneous linear wave equation has decay rate as*

$$|\partial_\alpha u| \leq C t^{1-n/2} T^{-1} = C t^{-(n-1)/2} (t-r+1)^{-1/2},$$

which is exactly the classical result.

3 Main result

3.1 Formalization of the problem and statement of the main result

One considers the following Cauchy problem associated to the quasilinear system:

$$(3.1) \quad \begin{cases} \square w_i + G_i^{j\alpha\beta}(w, \partial w) \partial_{\alpha\beta} w_j + D_i^2 w_i = F_i(w, \partial w), \\ w_i(B_0 + 1, x) = \varepsilon' w_{i0}, \\ \partial_t w_i(B_0 + 1, x) = \varepsilon' w_{i1}, \end{cases}$$

with D_i^2 constants, $D_i = 0$ for $1 \leq i \leq j_0$ and $D_i > 0$ for $j_0 + 1 \leq i \leq j_0 + k_0$. For simplicity one suppose that $D_i \geq 1$ for $j_0 + 1 \leq i \leq j_0 + k_0$, and

$$G_i^{j\alpha\beta}(w, \partial w) = A_i^{j\alpha\beta\gamma k} \partial_\gamma w_k + B_i^{j\alpha\beta k} w_k + O(|w| + |w'|)^2,$$

and

$$F_i(w, \partial w) = P_i^{\alpha\beta jk} \partial_\alpha w_j \partial_\beta w_k + Q_i^{\alpha jk} w_j \partial_\alpha w_k + R_i^{jk} w_j w_k + O(|w| + |w'|)^3.$$

Here $m^{\alpha\beta}$ are the coefficients of wave operator, and all $A_i^{j\alpha\beta\gamma k}, B_i^{j\alpha\beta k}, P_i^{\alpha\beta jk}, Q_i^{\alpha jk}$ and R_i^{jk} are constants with the absolute value controlled by K . Without lose of generality, one supposes $G_i^{j\alpha\beta} = G_i^{j\beta\alpha}$. To insure the hyperbolicity of the system, the following conditions of symmetry are imposed:

$$(3.2) \quad G_i^{j\alpha\beta} = G_j^{i\beta\alpha}.$$

For the convenience of proof, one makes the following convention of index: the Latin index i, j, k, l, \dots denote the positive entire number $1, 2, 3, \dots, k_0 + j_0$. The Greek index $\alpha, \beta, \gamma, \dots$ denote the nonnegative entire number $0, 1, 2, 3$. The Latin index with a circumflex accent above it such as \hat{j} denote the entire number $1, 2, \dots, j_0$ and the Latin index with a hacek on it such as \check{j} denote the entire number $j_0 + 1, j_0 + 2, \dots, j_0 + k_0$. One also denote by

$$u_{\hat{j}} := w_{\hat{j}}, \quad v_{\check{k}} := w_{\check{k}},$$

the different components of w_j .

One supposes that

$$(3.3) \quad B_i^{j\alpha\beta\hat{k}} = Q_i^{\alpha j\hat{k}} = R_i^{\hat{j}\hat{k}} = R_i^{\hat{k}} = 0.$$

One also suppose the weak null conditions:

$$(3.4) \quad \underline{A}_i^{\hat{j}000\hat{k}} = \underline{B}_i^{\hat{j}00\hat{k}} = \underline{P}_i^{00\hat{j}\hat{k}} = 0,$$

The initial data w_{i0}, w_{i1} are supposed to be regular functions compactly supported on the disc $|x| \leq B$. Then in general the following global-in-time existence holds

Theorem 3.1. *Suppose (3.4) holds. Then there exists an $\varepsilon_0 > 0$ such that for any $0 \leq \varepsilon' \leq \varepsilon_0$, the cauchy problem (3.1) has a unique global in time solution.*

Remark 3.2. • One improvement compared with [2] is that in theorem 3.1, the nonlinear part can have a term such as $\partial_\alpha w_i \partial_{tt}^2 w_i$. Further more, in the proof, the technical $L^\infty - L^\infty$ estimate will not be used.

- This result is also valid for the case where the initial data $w_{i0} \in H^7(\mathbb{R}^3)$ and $w_{i1} \in H^6(\mathbb{R}^3)$. Compared with [2], where one needs $w_{i0} \in H^{19}(\mathbb{R}^3)$ and $w_{i1} \in H^{18}(\mathbb{R}^3)$, this is also a improvement.
- The condition (3.4) and (3.3) are far from optimal. In general this method of proof can be applied to the case where the coefficients $A_i^{j\alpha\beta\gamma k}, B_i^{j\alpha\beta k}, P_i^{\alpha\beta jk}, Q_i^{\alpha jk}$ and R_i^{jk} are regular functions with certain increasing rates. But here one prefers to write a some-how restricted but short theorem. Readers may check the proof of lemma 3.5 and 3.6, 3.7, 3.8 to get weaker assumptions on coefficients.

3.2 Preparations

To prove this result we need some preparations. The following lemma is the principle of the so called bootstrap method:

Lemma 3.3 (Principle of continuity). *Let u be the regular local in time solution of the following quasilinear wave equation*

$$(3.5) \quad \begin{cases} \widetilde{\square}_{g(u, \partial u)} u = F(u, \partial u), \\ u(0, x) = u_0, \quad u_t(0, x) = u_1. \end{cases}$$

Then if the life span time T^ of u is finite, one has*

$$(3.6) \quad \sup_{\substack{0 \leq s < T^* \\ |I| \leq p(n), |J| \leq 2}} E_m(s, H^I \partial^J u) = \infty.$$

Proof. If (3.6) does not hold then from lemma 2.5

$$\sup_{\substack{0 \leq s < T^* \\ |J| \leq 2}} |\partial^J u| < \infty$$

By the theorem 6.4.11 of [1], one sees that $T^* = \infty$. □

A second result that one needs is a more technical energy estimated will be established, which is designed for the proof. One defines the following curved energy on hyperboloid H_s :

$$E_G(s, w_i) := E_m(s, w_i) + 2 \int_{H_s} (\partial_t w_i \partial_\beta w_j G_i^{j\alpha\beta}) \cdot (1, -x^a/t) dx - \int_{H_s} (\partial_\alpha w_i \partial_\beta w_j G_i^{j\alpha\beta}) dx.$$

The the following energy estimate holds:

Lemma 3.4 (Energy estimate). *Let $\{w_i\}$ a regular solution to the Cauchy problem (3.1). Suppose that the following estimates holds: If the following assumptions holds,*

$$(3.7) \quad \sum_i E_m(s, w_i) \leq 3 \sum_i E_G(s, w_i),$$

$$(3.8) \quad \int_{H_s} \left(\partial_\alpha G_i^{j\alpha\beta} \partial_t w_i \partial_\beta w_j - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_\alpha w_i \partial_\beta w_j \right) \frac{s}{t} dx \leq M_i(s) \sum_i E_m(s, w_i),$$

and

$$(3.9) \quad \left(\int_{H_s} |F_i|^2 dx \right)^{1/2} \leq L_i(s) + N_i(s) \sum_j E_m(s, w_j)^{1/2}.$$

Then the following energy estimate holds:

$$(3.10) \quad \begin{aligned} & \left(\sum_i E_m(s, w_i) \right)^{1/2} \\ & \leq \sqrt{3} \left(\sum_i E_G(B+1, w_i) \right)^{1/2} \exp \left(\int_{B+1}^s \sum_i \left(3M_i(\tau) + \sqrt{3(j_0 + k_0)} N_i(\tau) \right) d\tau \right) \\ & \quad + \int_{B+1}^s 3 \sum_i L_i(\tau) \exp \left(\int_{B+1}^\tau \sum_i \left(3M_i(\tau') + \sqrt{3(j_0 + k_0)} N_i(\tau') \right) d\tau' \right) d\tau. \end{aligned}$$

Proof. Under the assumptions (3.2), taking $\partial_t w_i$ as multiplier, the standard energy estimate procedure gives

$$\begin{aligned} & \sum_i \left(\frac{1}{2} \partial_t \sum_\alpha (\partial_\alpha w_i)^2 + \sum_a \partial_a (\partial_a w_i \partial_t w_i) + \partial_\alpha (G_i^{j\alpha\beta} \partial_t w_i \partial_\beta w_j) - \frac{1}{2} \partial_t (G_i^{j\alpha\beta} \partial_\alpha w_i \partial_\beta w_j G) \right) \\ & = \sum_i \partial_t w_i F_i + \sum_i \left(\partial_\alpha G_i^{j\alpha\beta} \partial_t w_i \partial_\beta w_j - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_\alpha w_i \partial_\beta w_j \right) \end{aligned}$$

Then integrate in the region G_{B+1}^s and use the Stokes formulae, one gets

$$\begin{aligned} \frac{1}{2} \sum_i (E_G(s, w_i) - E_G(B, w_i)) &= \int_{G_{B+1}^s} \partial_t w_i F_i dx + \sum_i \int_{G_{B+1}^s} \partial_\alpha G_i^{j\alpha\beta} \partial_t w_i \partial_\beta w_j \\ &\quad - \frac{1}{2} \partial_t G_i^{j\alpha\beta} \partial_\alpha w_i \partial_\beta w_j dx, \end{aligned}$$

which leads to

$$\begin{aligned} \frac{d}{ds} \sum_i E_G(s, w_i) &= 2 \sum_i \int_{H_s} (s/t) \partial_\alpha G_i^{j\alpha\beta} \partial_t w_i \partial_\beta w_j - (s/2t) \partial_t G_i^{j\alpha\beta} \partial_\alpha w_i \partial_\beta w_j dx \\ &\quad + 2 \int_{H_s} (s/t) \partial_t w_i F_i dx \end{aligned}$$

So one gets

$$\begin{aligned} &\left(\sum_i E_G(s, w_i) \right)^{1/2} \frac{d}{ds} \left(\sum_i E_G(s, w_i) \right)^{1/2} \\ &\leq \sqrt{3} \sum_i \left(\int_{H_s} |F_i|^2 dx \right)^{1/2} E_m(s, w_i)^{1/2} + \sum_i M_i(s) \sum_j E_m(s, w_j) \\ &\leq \sqrt{3} \left(\sum_i \int_{H_s} |F_i|^2 dx \right)^{1/2} \left(\sum_i E_G(s, w_i) \right)^{1/2} + 3 \sum_i M_i(s) \sum_j E_G(s, w_j), \end{aligned}$$

which leads to

$$\begin{aligned} &\frac{d}{ds} \left(\sum_i E_G(s, w_i) \right)^{1/2} \\ &\leq \sqrt{3} \left(\sum_i \int_{H_s} |F_i|^2 dx \right)^{1/2} + 3 \sum_i M_i(s) \left(\sum_j E_G(s, w_j) \right)^{1/2} \\ &\leq \sqrt{3} \sum_i \left(L_i(s) + N_i(s) \sum_j E_G(s, w_j)^{1/2} \right) + 3 \sum_i M_i(s) \left(\sum_j E_G(s, w_j) \right)^{1/2} \\ &\leq \sqrt{3} \sum_i L_i(s) + \sqrt{3(j_0 + k_0)} \sum_i N_i(s) \left(\sum_j E_G(s, w_j) \right)^{1/2} + 3 \sum_i M_i(s) \left(\sum_j E_G(s, w_j) \right)^{1/2}. \end{aligned}$$

By Gronwall's lemma, (3.10) is proved. \square

3.3 Proof of the main result

proof of theorem 3.1. For any $\varepsilon, C_1 > 0$, by theorem A.1, there exists an $\varepsilon_0(B) > 0$ such that for any $0 \leq \varepsilon' \leq \varepsilon_0(B)$, $E_m(B+1, Z^I w_i) \leq \varepsilon C_1$. Then one uses the continuity method. Suppose that on a interval $[B+1, T)$, the energy $E_m(s, Z^I w_i)$ satisfy

$$\begin{aligned} (3.11) \quad &\sum_{|I^*| \leq 7} \left(\sum_i E_m(s, Z^I w_i) \right)^{1/2} \leq C_1 \varepsilon s^\delta, \\ &E_m(s, Z^I u_{\hat{i}}) \leq C_1 \varepsilon, \quad \text{for } 1 \leq \hat{i} \leq j_0. \quad |I| \leq 5. \end{aligned}$$

where $0 < \delta \leq 1/6$. By lemma 2.3, one has the following L^2 estimates:

$$\begin{aligned}
(3.12) \quad & \sum_{\substack{\hat{i}, \alpha \\ |I| \leq 5}} \left(\int_{H_s} |(s/t) Z^I \partial_\alpha u_{\hat{i}}|^2 dx \right)^{1/2} + \sum_{\substack{\hat{i}, \alpha \\ |I| \leq 5}} \left(\int_{H_s \cap \{r \geq r/2\}} |Z^I \bar{\partial}_a u_{\hat{i}}|^2 dx \right)^{1/2} \leq C(n) C_1 \varepsilon, \\
& \sum_{\substack{\hat{i}, \alpha \\ |I^*| \leq 7}} \left(\int_{H_s} |(s/t) Z^{I^*} \partial_\alpha w_{\hat{i}}|^2 dx \right)^{1/2} + \sum_{\substack{\hat{i}, \alpha \\ |I^*| \leq 7}} \left(\int_{H_s \cap \{r \geq r/2\}} |Z^{I^*} \bar{\partial}_a w_{\hat{i}}|^2 dx \right)^{1/2} \leq C(n) C_1 \varepsilon s^\delta, \\
& \sum_{\substack{\hat{j} \\ |I^*| \leq 7}} \left(\int_{H_s} |Z^{I^*} v_{\hat{j}}|^2 dx \right)^{1/2} \leq C(n) C_1 \varepsilon s^\delta.
\end{aligned}$$

Also, by lemma 2.6 and lemma ??, one has, for $|J| \leq 3$ and $|J^*| \leq 5$,

$$\begin{aligned}
(3.13) \quad & \sup_{H_s} |st^{1/2} \partial_\alpha Z^J u_{\hat{j}}| + \sup_{H_s \cap \{r \geq t/2\}} |t^{3/2} \bar{\partial}_a Z^J u_{\hat{j}}| \leq C(n) C_1 \varepsilon, \\
& \sup_{H_s} \left(|st^{1/2} \partial_\alpha Z^{J^*} v_{\hat{k}}| + |t^{3/2} Z^{J^*} v_{\hat{k}}| \right) + \sup_{H_s \cap \{r \geq t/2\}} |t^{3/2} \bar{\partial}_a Z^{J^*} v_{\hat{k}}| \leq C(n) C_1 \varepsilon s^\delta.
\end{aligned}$$

From lemma 2.3, one gets

$$\begin{aligned}
(3.14) \quad & \sup_{H_s} |st^{1/2} Z^J \partial_\alpha u_{\hat{i}}| + \sup_{H_s \cap \{r \geq t/2\}} |t^{3/2} Z^J \bar{\partial}_a u_{\hat{i}}| \leq C(n) C_1 \varepsilon, \\
& \sup_{H_s} \left(|st^{1/2} Z^{J^*} \partial_\alpha v_{\hat{k}}| + |t^{3/2} Z^{J^*} v_{\hat{k}}| \right) + \sup_{H_s \cap \{r \geq t/2\}} |t^{3/2} Z^{J^*} \bar{\partial}_a v_{\hat{k}}| \leq C(n) C_1 \varepsilon s^\delta.
\end{aligned}$$

One derives the equation (3.1) with respect to a product Z^I , and gets

$$(3.15) \quad \square Z^I w_i + G_i^{j\alpha\beta} \partial_{\alpha\beta} Z^I w_j + D_i^2 Z^I w_i = -[Z^I, G_i^{j\alpha\beta} \partial_{\alpha\beta}] w_j + Z^I F_i(w, w').$$

One also writes the first j_0 equations:

$$(3.16) \quad \square Z^I u_{\hat{j}} = Z^I F_{\hat{j}} - Z^I (G_{\hat{j}}^{k\alpha\beta} \partial_{\alpha\beta} w_k).$$

For technical reason, when $|I| = 7$ and $|J| = 6$, the following system will also be considered.

$$(3.17) \quad \begin{cases} \square Z^I w_i + G_i^{j\alpha\beta} \partial_{\alpha\beta} Z^I w_j + D_i^2 Z^I w_i = -[Z^I, G_i^{j\alpha\beta} \partial_{\alpha\beta}] w_j + Z^I F_i(w, w'). \\ \square Z^J w_i + G_i^{j\alpha\beta} \partial_{\alpha\beta} Z^J w_j + D_i^2 Z^J w_i = -[Z^J, G_i^{j\alpha\beta} \partial_{\alpha\beta}] w_j + Z^J F_i(w, w'). \end{cases}$$

Then by (3.15), the energy estimate (2.3) gives for any $|I| \leq 5$ and $1 \leq \hat{j} \leq j_0$,

$$E_m(s, Z^I u_{\hat{i}})^{1/2} \leq E_m(s, Z^I u_{\hat{i}})^{1/2} + \int_{B+1}^s \left(\int_{H_\tau} |Z^I F_{\hat{i}} - Z^I (G_{\hat{i}}^{j\alpha\beta} w_j)|^2 dx \right)^{1/2} d\tau.$$

By (3.17), the energy estimate (3.10) gives,

$$\begin{aligned}
& \left(\sum_{\substack{\hat{i} \\ 6 \leq |I^*| \leq 7}} E_m(s, Z^{I^*} w_{\hat{i}}) \right)^{1/2} \\
& \leq \sqrt{3} \left(\sum_{\substack{\hat{i} \\ 6 \leq |I^*| \leq 7}} E_m(B+1, Z^{I^*} w_{\hat{i}}) \right)^{1/2} \exp \left(\int_{B+1}^s \sum_{\hat{i}} (3M_i(\tau) + \sqrt{3(j_0 + k_0)} N_i(\tau)) d\tau \right) \\
& \quad + \int_{B+1}^s 3 \sum_{\hat{i}} L_i(\tau) \exp \left(\int_{B+1}^s \sum_{\hat{i}} (3M_i(\tau') + \sqrt{3(j_0 + k_0)} N_i(\tau')) d\tau' \right) d\tau.
\end{aligned}$$

where

$$\left(\int_{H_s} |Z^{I^*} F_i - [Z^{I^*}, G_i^{j\alpha\beta} \partial_{\alpha\beta}] w_j|^2 dx \right) \leq L_i(s) + N_i(s) \sum_{6 \leq |I| \leq 7} E_m(s, Z^I w_i)^{1/2}$$

and

$$\int_{H_s} \frac{s}{t} \left((\partial_\alpha G_i^{j\alpha\beta}) \partial_t Z^{I^*} w_i \partial_\beta Z^{I^*} w_j - \frac{1}{2} (\partial_t G_i^{j\alpha\beta}) \partial_\alpha Z^{I^*} w_i \partial_\beta Z^{I^*} w_i \right) dx \leq M_i(s) \sum_i E_m(s, Z^{I^*} w_i)$$

And By (3.16), for any $|I| \leq 5$,

$$\begin{aligned} \left(\sum_i E_m(s, Z^I w_i) \right)^{1/2} &\leq \sqrt{3} \left(\sum_i E_m(B+1, Z^I w_i) \right)^{1/2} \exp \left(\int_{B+1}^s \sum_i 3M_i(\tau) d\tau \right) \\ &\quad + \int_{B+1}^s 3 \sum_i L_i(\tau) \exp \left(\int_{B+1}^s \sum_i 3M_i(\tau') d\tau' \right) d\tau, \end{aligned}$$

where

$$\left(\int_{H_s} |Z^I F_i - [Z^I, G_i^{j\alpha\beta} \partial_{\alpha\beta}] w_j|^2 dx \right) \leq L_i(s)$$

and

$$\int_{H_s} \frac{s}{t} \left((\partial_\alpha G_i^{j\alpha\beta}) \partial_t Z^I w_i \partial_\beta Z^I w_j - \frac{1}{2} (\partial_t G_i^{j\alpha\beta}) \partial_\alpha Z^I w_i \partial_\beta Z^I w_i \right) dx \leq M_i(s) \sum_i E_m(s, Z^I w_i)$$

Suppose the following estimates can be deduced from (3.3), (3.4), (3.12), (3.13) and (3.14):
For any $|I| \leq 5$,

$$\begin{aligned} (3.18) \quad \left(\sum_{\hat{j}} \int_{H_\tau} |Z^I F_i - Z^I (G_i^{j\alpha\beta} w_j)|^2 dx \right)^{1/2} &\leq C(n) K(C_1 \varepsilon)^2 \tau^{-1-\theta} \\ &= L_{\hat{j}}(s) \end{aligned}$$

with

$$\int_{B+1}^\infty L_{\hat{j}} ds = C(n) K(C_1 \varepsilon)^2 \theta^{-1} (B+1)^{-\theta} = L < \infty.$$

And for any $|I^*| \leq 7$ (if $|I^*| \leq 5$ then $N_i = 0$):

$$\begin{aligned} (3.19) \quad &\left(\int_{H_\tau} |[G_i^{j\alpha\beta} \partial_{\alpha\beta}, Z^{I^*}] w_j + Z^{I^*} F_i(w, w')|^2 dx \right)^{1/2} \\ &\leq C(n) K(C_1 \varepsilon)^2 \tau^{-1} + C(n) K C_1 \varepsilon \tau^{-1} \sum_{6 \leq |I| \leq |I^*|} E_m(\tau, Z^{I^*} w_j)^{1/2} \\ &= L_i(s) + N_i(s) \sum_{6 \leq |I| \leq |I^*|} E_m(\tau, Z^{I^*} w_i)^{1/2}, \end{aligned}$$

$$(3.20) \quad M_i(\tau) = C(n) K C_1 \varepsilon \tau^{-1}.$$

Then, for $|I| \leq 6$

$$E_m(s, Z^I u_{\hat{j}})^{1/2} \leq E_m(s, Z^I u_{\hat{j}})^{1/2} + C(n) K(C_1 \varepsilon)^2 \theta^{-1} (B+1)^{-\theta}.$$

$$\left(\sum_i E_m(s, Z^I w_i) \right)^{1/2} \leq (\sqrt{3} C_0 + C_1) (s/B + 1)^{C(j_0, k_0) K C_1 \varepsilon}.$$

Similarly, for $6 \leq |I^*| \leq 7$,

$$\left(\sum_i E_m(s, Z^{I^*} w_i) \right)^{1/2} \leq (\sqrt{3}C_0 + C_1)(s/B + 1)^{C(j_0, k_0)KC_1\varepsilon}.$$

Now consider

$$T^*(\varepsilon) := \sup_T (\text{for any } B + 1 \leq s \leq T, (3.12) \text{ holds}).$$

By continuity, at least one of the following two equations holds:

$$\begin{aligned} \left(\sum_{|I^*| \leq 7} E_m(s, Z^I w_i) \right)^{1/2} &= C_1 \varepsilon s^\delta, \\ E_m(s, Z^I u_{\hat{i}})^{1/2} &= C_1 \varepsilon, \quad \text{for } 1 \leq \hat{i} \leq j_0. \end{aligned}$$

When $KC(j_0, k_0)C_1\varepsilon = \delta$, $C_1 \geq 2C_0$ and $(B + 1)^\delta > 2$

$$\left(\sum_{|I^*| \leq 9} E_m(s, Z^I w_i) \right)^{1/2} < C_1 \varepsilon s^\delta.$$

When $KC(j_0, k_0)C_1\varepsilon = \delta$, $C_1 \geq 2C_0$ and $(B + 1)^\theta > 2\delta^2\theta^{-1}(KC(j_0, k_0))^{-1}$,

$$E_m(s, Z^I u_{\hat{i}})^{1/2} < C_1 \varepsilon, \quad \text{for } 1 \leq \hat{i} \leq j_0.$$

So for ε sufficiently small, C_1 and B sufficiently large, on time interval $[B + 1, \infty)$, (3.12) holds. Then lemma 3.3 completes the proof. \square

The remained work is to verify (3.7) and (3.18) - (3.20) under the assumption of (3.3), (3.4), (3.12) and (3.14).

The following lemma is to guarantee (3.7).

Lemma 3.5. *Suppose (3.3) and (3.13) hold. Then following estimate holds*

$$\sum_i E_g(s, Z^I w_i) \leq 3 \sum_i E_m(s, Z^I w_i).$$

Proof. One notice that

$$\sum_{i,j,\alpha,\beta} |G_i^{j\alpha\beta}| \leq CK \sum_i (|\partial w_i| + |w_i|).$$

Then by simple calculation

$$\begin{aligned} \sum_i |E_G(s, w_i) - E_m(s, w_i)| &= \left| 2 \int_{H_s} (\partial_t w_i \partial_\beta w_j G_i^{j\alpha\beta}) \cdot (1, -x^a/t) dx - \int_{H_s} (\partial_\alpha w_i \partial_\beta w_j G_i^{j\alpha\beta}) dx \right| \\ &\leq 2 \int_{H_s} \left(\sum_{i,j,\alpha,\beta} |G_i^{j\alpha\beta}| \right) \cdot \left(\sum_{\alpha,k} |\partial_\alpha w_k|^2 \right) dx \\ &\leq 2CK \int_{H_s} \sum_i (|\partial w_i| + |w_i|) \cdot \left(\sum_{\alpha,k} |\partial_\alpha w_k|^2 \right) dx \\ &\leq 2CKC(n)C_1\varepsilon \int_{H_s} (t^{-3/2}s^\delta + t^{-1/2}s^{-1})(t/s)^2 \cdot \left(\sum_{\alpha,k} |(s/t)\partial_\alpha w_k|^2 \right) dx \\ &= 2CKC(n)C_1\varepsilon \int_{H_s} (t^{1/2}s^{-2+\delta} + t^{3/2}s^{-3}) \cdot \left(\sum_{\alpha,k} |(s/t)\partial_\alpha w_k|^2 \right) dx \\ &\leq 2C\delta \sum_i E_m(s, w_i). \end{aligned}$$

Here one takes $KC_1C(n)\varepsilon \leq \delta$. When $0 < \delta$ small enough,

$$\sum_i |E_G(s, w_i) - E_m(s, w_i)| \leq 2 \sum_i E_m(s, w_i),$$

which complete the proof. \square

All of the following lemmas are L^2 type estimates and their proofs are similar. The only simple idea used is to subtitle the decay estimates (3.14) and the energy assumption (3.12) into the expression. The calculation is long and tedious and will be left in Appendix B.

The first lemma is to guarantee (3.18).

Lemma 3.6. *Suppose that (3.3), (3.4) and (3.14) hold, Then for any $|I| \leq 5$,*

$$(3.21) \quad \left(\int_{H_s} |Z^I F_i(w, w')|^2 dx \right)^{1/2} \leq C(n)(C_1\varepsilon)^2 K s^{-3/2+2\delta},$$

and

$$(3.22) \quad \left(\int_{H_s} |Z^I (G_i^{j\alpha\beta} \partial_{\alpha\beta} w_j)|^2 dx \right)^{1/2} \leq C(n)(C_1\varepsilon)^2 K s^{-3/2+2\delta}.$$

The last lemma is to guarantee (3.19)

Lemma 3.7. *Suppose (3.3), (3.4), (3.12) and (3.14) hold, then the following estimates hold for any $|I^*| \leq 7$:*

$$(3.23) \quad \left(\int_{H_s} |Z^{I^*} F_i(w, w')|^2 dx \right)^{1/2} \leq C(n)(C_1\varepsilon)^2 K s^{-1} + C(n)C_1\varepsilon K s^{-1} \sum_{6 \leq |I| \leq |I^*|} E_m(s, Z^I w_j),$$

$$(3.24) \quad \left(\int_{H_s} |[Z^{I^*}, G_i^{j\alpha\beta} \partial_{\alpha\beta}] w_j|^2 dx \right)^{1/2} \leq C(n)(C_1\varepsilon)^2 K s^{-1} + C(n)C_1\varepsilon K s^{-1} \sum_{6 \leq |I| \leq |I^*|} E_m(s, Z^I w_j),$$

When $|I^*| \leq 5$ the last terms disappear.

The following lemma is to guarantee (3.20):

Lemma 3.8. *Suppose (3.3), (3.12) and (3.14) hold, then for any $|I^*| \leq 7$ the following estimates is true:*

$$(3.25) \quad \int_{H_s} \frac{s}{t} \left((\partial_\alpha G_i^{j\alpha\beta}) \partial_t Z^{I^*} w_i \partial_\beta Z^{I^*} w_j - \frac{1}{2} (\partial_t G_i^{j\alpha\beta}) \partial_\alpha Z^{I^*} w_i \partial_\beta Z^{I^*} w_j \right) dx \leq M_i(s) \sum_k E_m(s, Z^I w_k),$$

where

$$M_i(s) = C(n)C_1\varepsilon K s^{-1}.$$

Proof of lemma 3.6. One will firstly prove (3.21). By (3.3), for any $|I| \leq 5$,

$$F_i = P_i^{\alpha\beta jk} \partial_\alpha w_j \partial_\beta w_k + Q_i^{\alpha j \tilde{k}} \partial_\alpha w_j v_{\tilde{k}} + R_i^{j \tilde{k}} v_j v_{\tilde{k}}.$$

For the first term:

$$\begin{aligned} P_i^{\alpha\beta jk} \partial_\alpha w_j \partial_\beta w_k &= P_i^{\alpha\beta \hat{j} \hat{k}} \partial_\alpha u_{\hat{j}} \partial_\beta u_{\hat{k}} + P_i^{\alpha\beta \hat{j} \tilde{k}} \partial_\alpha u_{\hat{j}} \partial_\beta v_{\tilde{k}} \\ &\quad P_i^{\alpha\beta \tilde{j} \hat{k}} \partial_\alpha v_{\tilde{j}} \partial_\beta u_{\hat{k}} + P_i^{\alpha\beta \tilde{j} \tilde{k}} \partial_\alpha v_{\tilde{j}} \partial_\beta v_{\tilde{k}} \\ &=: Y_1 + Y_2 + Y_3 + Y_4 \end{aligned}$$

Now consider Y_1 .

$$\begin{aligned} \left(\int_{H_s} |Z^I Y_1|^2 dx \right)^{1/2} &\leq \left(\int_{H_s \cap \{r \leq t/2\}} |Z^I Y_1|^2 dx \right)^{1/2} + \left(\int_{H_s \cap \{r \geq t/2\}} |Z^I Y_1|^2 dx \right)^{1/2} \\ &=: S_1 + S_2. \end{aligned}$$

$$\begin{aligned} S_1 &\leq \left(\int_{H_s \cap \{r \leq t/2\}} |Z^I (P_i^{\alpha\beta\hat{j}\hat{k}} \partial_\alpha u_{\hat{j}} \partial_\beta u_{\hat{k}})|^2 dx \right)^{1/2} \\ &\leq \sum_{I_1+I_2=I} \left(\int_{H_s \cap \{r \leq t/2\}} |P_i^{\alpha\beta\hat{j}\hat{k}} Z^{I_1} \partial_\alpha u_{\hat{j}} Z^{I_2} \partial_\beta u_{\hat{k}}|^2 dx \right)^{1/2} \\ &\leq \sum_{\substack{|I_1| \leq 2 \\ I_1+I_2=I}} \left(\int_{H_s \cap \{r \leq t/2\}} |P_i^{\alpha\beta\hat{j}\hat{k}} Z^{I_1} \partial_\alpha u_{\hat{j}} Z^{I_2} \partial_\beta u_{\hat{k}}|^2 dx \right)^{1/2} \\ &\quad + \sum_{\substack{|I_2| \leq 2 \\ I_1+I_2=I}} \left(\int_{H_s \cap \{r \leq t/2\}} |P_i^{\alpha\beta\hat{j}\hat{k}} Z^{I_1} \partial_\alpha u_{\hat{j}} Z^{I_2} \partial_\beta u_{\hat{k}}|^2 dx \right)^{1/2} \\ &\leq \sum_{\substack{|I_1| \leq 2 \\ I_1+I_2=I}} \left(\int_{H_s \cap \{r \leq t/2\}} |KC(n) C_1 \varepsilon t^{-1/2} s^{-1}(t/s)|^2 \cdot |(s/t) Z^{I_2} \partial_\beta u_{\hat{k}}|^2 dx \right)^{1/2} \\ &\quad + \sum_{\substack{|I_2| \leq 2 \\ I_1+I_2=I}} \left(\int_{H_s \cap \{r \leq t/2\}} |(s/t) Z^{I_1} \partial_\alpha u_{\hat{j}}|^2 \cdot |KC(n) C_1 \varepsilon t^{-1/2} s^{-1}(t/s)|^2 dx \right)^{1/2} \\ &\leq C(n) (C_1 \varepsilon)^2 K s^{-3/2}. \end{aligned}$$

To estimate S_2 , one uses the “one-one” frame and the weak null conditions (3.4).

$$\begin{aligned} S_2 &= \left(\int_{H_s \cap \{r \geq t/2\}} |Z^I (\underline{P}_i^{\alpha\beta\hat{j}\hat{k}} \underline{\partial}_\alpha u_{\hat{j}} \underline{\partial}_\beta u_{\hat{k}})|^2 dx \right)^{1/2} \\ &\leq \left(\int_{H_s \cap \{r \geq t/2\}} |Z^I (\underline{P}_i^{a\beta\hat{j}\hat{k}} \underline{\partial}_a u_{\hat{j}} \underline{\partial}_\beta u_{\hat{k}})|^2 dx \right)^{1/2} \\ &\quad + \left(\int_{H_s \cap \{r \geq t/2\}} |Z^I (\underline{P}_i^{\alpha b\hat{j}\hat{k}} \underline{\partial}_\alpha u_{\hat{j}} \underline{\partial}_b u_{\hat{k}})|^2 dx \right)^{1/2} \\ &=: S_2^{(1)} + S_2^{(2)}. \end{aligned} \tag{3.26}$$

By lemma 2.4,

$$\begin{aligned} S_2^{(1)} &\leq \sum_{I_1+I_2+I_3=I} \left(\int_{H_s \cap \{r \geq t/2\}} |Z^{I_3} \underline{P}_i^{a\beta\hat{j}\hat{k}} Z^{I_1} \underline{\partial}_a u_{\hat{j}} Z^{I_2} \underline{\partial}_\beta u_{\hat{k}}|^2 dx \right)^{1/2} \\ &\leq \sum_{\substack{|I_1| \leq 2 \\ I_1+I_2+I_3=I}} \left(\int_{H_s \cap \{r \geq t/2\}} |Z^{I_3} \underline{P}_i^{a\beta\hat{j}\hat{k}} Z^{I_1} \underline{\partial}_a u_{\hat{j}} Z^{I_2} \underline{\partial}_\beta u_{\hat{k}}|^2 dx \right)^{1/2} \\ &\quad + \sum_{\substack{|I_2| \leq 2 \\ I_1+I_2+I_3=I}} \left(\int_{H_s \cap \{r \geq t/2\}} |Z^{I_3} \underline{P}_i^{a\beta\hat{j}\hat{k}} Z^{I_1} \underline{\partial}_a u_{\hat{j}} Z^{I_2} \underline{\partial}_\beta u_{\hat{k}}|^2 dx \right)^{1/2} \\ &\leq C(n) (C_1 \varepsilon)^2 K s^{-3/2}. \end{aligned} \tag{3.27}$$

$S_2^{(2)}$ is estimated in the same way. Then

$$\left(\int_{H_s} |Z^I Y_1|^2 \right)^{1/2} \leq C(n) (C_1 \varepsilon)^2 s^{-3/2}.$$

Now consider the term of Y_2 , and Y_3 . One notices that by (3.12), for any $|I| \leq 6$ and $|J| \leq 5$,

$$(3.28) \quad \begin{aligned} \left(\int_{H_s} |Z^I \partial_\alpha v_{\tilde{k}}|^2 dx \right)^{1/2} &\leq C_1 \varepsilon, \\ \left(\int_{H_s} |Z^J \partial_{\alpha\beta} v_{\tilde{k}}|^2 dx \right)^{1/2} &\leq C_1 \varepsilon. \end{aligned}$$

Taking this into account, one simply substitutes (3.12) and (3.14) into the expression of Y_2 and Y_3 (null conditions are not imposed here) and gets

$$\left(\int_{H_s} |Z^I Y_2|^2 dx \right)^{1/2} + \left(\int_{H_s} |Z^I Y_3|^2 dx \right)^{1/2} \leq C(n)(C_1 \varepsilon)^{1/2} s^{-3/2+\delta}$$

The estimate on Y_4 is even simpler than that of Y_2 and Y_3 . One simply substitutes (3.12) and (3.14) into the expression, and gets

$$\left(\int_{H_s} |Z^I Y_4|^2 dx \right)^{1/2} \leq C(n)(C_1 \varepsilon)^2 s^{-3/2+2\delta}.$$

The estimate of the integrals

$$\int_{H_s} |Z^I (Q_i^{\alpha j \tilde{k}} \partial_\alpha w_j v_{\tilde{k}})|^2 dx$$

and

$$\int_{H_s} |Z^I (R_i^{j \tilde{k}} v_j v_{\tilde{k}})|^2 dx,$$

are just substitutions of (3.12) and (3.14). One gets

$$\int_{H_s} |Z^I (Q_i^{\alpha j \tilde{k}} \partial_\alpha w_j v_{\tilde{k}})|^2 dx \leq C(n)(C_1 \varepsilon)^2 K s^{-3/2+2\delta},$$

and

$$\int_{H_s} |Z^I (R_i^{j \tilde{k}} v_j v_{\tilde{k}})|^2 dx \leq C(n)(C_1 \varepsilon)^2 K s^{-3/2+2\delta}.$$

So one gets for any $|I| \leq 8$,

$$\left(\int_{H_s} |Z^I F_i|^2 \right)^{1/2} \leq C(n)(C_1 \varepsilon)^2 s^{-3/2+2\delta}.$$

The proof of (3.22) is quite similar.

$$Z^I G_i^{j\alpha\beta} \partial_{\alpha\beta} w_j = Z^I G_i^{j\alpha\beta} \partial_{\alpha\beta} u_{\hat{j}} + Z^I G_i^{j\alpha\beta} \partial_{\alpha\beta} v_{\tilde{j}}.$$

The first term is decomposed into three pieces:

$$\begin{aligned} Z^I G_i^{j\alpha\beta} \partial_{\alpha\beta} u_{\hat{j}} &= Z^I (A_i^{j\alpha\beta\gamma\tilde{k}} \partial_\gamma u_{\tilde{k}} \partial_{\alpha\beta} u_{\hat{j}}) + Z^I (A_i^{j\alpha\beta\gamma\tilde{k}} \partial_\gamma v_{\tilde{k}} \partial_{\alpha\beta} u_{\hat{j}}) + Z^I (B_i^{j\alpha\beta\tilde{k}} v_{\tilde{k}} \partial_{\alpha\beta} u_{\hat{j}}) \\ &=: M_1(s) + M_2(s) + M_3(s). \end{aligned}$$

The estimate on M_2 is simple.

$$\begin{aligned}
M_2(s) &\leq \left(\int_{H_s} |Z^I (A_i^{\hat{j}\alpha\beta\gamma\hat{k}} \partial_\gamma v_{\hat{k}} \partial_{\alpha\beta} u_{\hat{j}})|^2 dx \right)^{1/2} \\
&\leq \sum_{I_1+I_2=I} \left(\int_{H_s} K^2 |Z^{I_1} \partial_\gamma v_{\hat{k}}|^2 \cdot |Z^{I_2} \partial_{\alpha\beta} u_{\hat{j}}|^2 dx \right)^{1/2} \\
&\leq \sum_{\substack{|I_1| \leq 2 \\ I_1+I_2=I}} \left(\int_{H_s} K^2 |Z^{I_1} \partial_\gamma v_{\hat{k}}|^2 \cdot |Z^{I_2} \partial_{\alpha\beta} u_{\hat{j}}|^2 dx \right)^{1/2} \\
&\quad + \sum_{\substack{|I_2| \leq 2 \\ I_1+I_2=I}} \left(\int_{H_s} K^2 |Z^{I_1} \partial_\gamma v_{\hat{k}}|^2 \cdot |Z^{I_2} \partial_{\alpha\beta} u_{\hat{j}}|^2 dx \right)^{1/2} \\
&\leq KC(n)C_1\varepsilon \sum_{|I_2| \leq 5} \left(\int_{H_s} (t^{-3/2} s^\delta (t/s))^2 \cdot |(s/t) Z^{I_2} \partial_{\alpha\beta} u_{\hat{j}}|^2 dx \right)^{1/2} \\
&\quad + KC(n)C_1\varepsilon \sum_{|I_1| \leq 5} \left(\int_{H_s} |Z^{I_1} \partial_\gamma v_{\hat{k}}|^2 (t^{-1/2} s^{-1})^2 dx \right)^{1/2} \\
&\leq KC(n)(C_1\varepsilon)^2 s^{-3/2+2\delta}.
\end{aligned}$$

Similarly

$$M_3 \leq C(n)(C_1\varepsilon)^2 K s^{-3/2+2\delta}.$$

The estimate of $M_1(s)$ is the more difficult than the others.

$$\begin{aligned}
M_1(s) &\leq \left(\int_{H_s \cap \{r \leq t/2\}} |Z^I (A_i^{\hat{j}\alpha\beta\gamma\hat{k}} \partial_\gamma u_{\hat{k}} \partial_{\alpha\beta} u_{\hat{j}})|^2 \right)^{1/2} \\
&\quad + \left(\int_{H_s \cap \{r \leq t/2\}} |Z^I (A_i^{\hat{j}\alpha\beta\gamma\hat{k}} \partial_\gamma u_{\hat{k}} \partial_{\alpha\beta} u_{\hat{j}})|^2 \right)^{1/2} \\
&=: S_1 + S_2 \\
S_1 &\leq \sum_{\substack{|I_1| \leq 2 \\ I_1+I_2=I}} \left(\int_{H_s \cap \{r \leq t/2\}} K^2 |Z^{I_1} \partial_\gamma u_{\hat{k}}|^2 \cdot |Z^{I_2} \partial_{\alpha\beta} u_{\hat{j}}|^2 \right)^{1/2} \\
&\quad + \sum_{\substack{|I_2| \leq 2 \\ I_1+I_2=I}} \left(\int_{H_s \cap \{r \leq t/2\}} K^2 |Z^{I_1} \partial_\gamma u_{\hat{k}}|^2 \cdot |Z^{I_2} \partial_{\alpha\beta} u_{\hat{j}}|^2 \right)^{1/2} \\
&\leq C(n)(C_1\varepsilon)^2 s^{-3/2+\delta}. \\
S_2 &\leq \left(\int_{H_s \cap \{r \geq t/2\}} |Z^I (\underline{A}_i^{\hat{j}\alpha\beta\gamma\hat{k}} \underline{\partial}_\gamma u_{\hat{k}} \underline{\partial}_{\alpha\beta} u_{\hat{j}})|^2 \right)^{1/2} \\
&\quad - \left(\int_{H_s \cap \{r \geq t/2\}} |Z^I (\underline{A}_i^{\hat{j}\alpha\beta\gamma\hat{k}} \underline{\partial}_\gamma u_{\hat{k}} \underline{\partial}_\alpha (\Phi_\beta^{\beta'}) \partial_{\beta'} u_{\hat{j}})|^2 \right)^{1/2} \\
&:= H_1(s) + H_2(s)
\end{aligned}$$

The estimate on $H_2(s)$ is simple. One notice that $\underline{\partial}_\alpha \Phi_\beta^{\beta'} \leq Ct^{-1}$ when $r \geq t/2$. Then one gets

$$H_2(S) \leq C(n)(C_1\varepsilon)^2 K s^{-5/2}.$$

The estimate of $H_1(s)$ will consult the weak null conditions (3.4). Just as one as shown in (3.26) and (3.27),

$$H_1(s) \leq C(n)(C_1\varepsilon)^2 K s^{-3/2+\delta}.$$

To estimate the term

$$\int_{H_s} |Z^I (G_i^{\check{\alpha}\beta} \partial_{\alpha\beta} v_{\check{j}})|^2 dx,$$

One notices that

$$(3.29) \quad |Z^I G_i^{\check{\alpha}\beta}| \leq C(n)K \sum_{\substack{\alpha \\ |I'| \leq |I|}} \left(\sum_{\check{k}} Z^{I'} \partial_{\alpha} u_{\check{k}} + \sum_{\check{l}} Z^{I'} \partial_{\alpha} v_{\check{l}} + \sum_{\check{i}} Z^{I'} v_{\check{i}} \right).$$

So one gets

$$\begin{aligned} & \left(\int_{H_s} |Z^I (G_i^{\check{\alpha}\beta} \partial_{\alpha\beta} v_{\check{j}})|^2 dx \right)^{1/2} \\ & \leq \sum_{|I_1| \leq 2} \left(\int_{H_s} |Z^{I_1} G_i^{\check{\alpha}\beta} Z^{I_2} \partial_{\alpha\beta} v_{\check{j}}|^2 dx \right)^{1/2} \\ & \quad + \sum_{|I_2| \leq 2} \left(\int_{H_s} |Z^{I_1} G_i^{\check{\alpha}\beta} Z^{I_2} \partial_{\alpha\beta} v_{\check{j}}|^2 dx \right)^{1/2} \\ & \leq C(n)C_1 \varepsilon K \left(\int_{H_s} (t^{-1/2} s^{-1} + t^{-3/2} s^{\delta})^2 \cdot |Z^{I_2} \partial_{\alpha\beta} v_{\check{j}}|^2 dx \right)^{1/2} \\ & \quad + C(n)C_1 \varepsilon K \left(\int_{H_s} |Z^{I_1} G_i^{\check{\alpha}\beta}|^2 \cdot (t^{-3/2} s^{\delta})^2 dx \right)^{1/2} \\ & \leq C(n)(C_1 \varepsilon)^2 K s^{-3/2+2\delta}. \end{aligned}$$

□

Proof of lemma 3.7. One will firstly prove (3.23).

$$F_i = P_i^{\alpha\beta j\check{k}} \partial_{\alpha\beta} w_j \partial_{\beta} w_{\check{k}} + Q_i^{\alpha j\check{k}} \partial_{\alpha} w_j v_{\check{k}} + R_i^{\check{k}} v_{\check{j}} v_{\check{k}}.$$

And

$$\begin{aligned} P_i^{\alpha\beta j\check{k}} \partial_{\alpha\beta} w_j \partial_{\beta} w_{\check{k}} &= P_i^{\alpha\beta \hat{j}\hat{k}} \partial_{\alpha\beta} u_{\hat{j}} \partial_{\beta} u_{\check{k}} + P_i^{\alpha\beta \hat{j}\check{k}} \partial_{\alpha\beta} u_{\hat{j}} \partial_{\beta} v_{\check{k}} \\ &\quad + P_i^{\alpha\beta \hat{j}\hat{k}} \partial_{\alpha\beta} v_{\hat{j}} \partial_{\beta} u_{\check{k}} + P_i^{\alpha\beta \hat{j}\check{k}} \partial_{\alpha\beta} v_{\hat{j}} \partial_{\beta} v_{\check{k}}. \end{aligned}$$

Then the following estimates hold. Here $|I^*| \leq 8$. For the first term:

$$\begin{aligned} & \left(\int_{H_s} |Z^{I^*} (P_i^{\alpha\beta \hat{j}\check{k}} \partial_{\alpha} u_{\hat{j}} \partial_{\beta} u_{\check{k}})|^2 dx \right)^{1/2} \\ & \leq K \sum_{\substack{I_1^* + I_2^* = I^* \\ |I_1^*| \leq 3, |I_2^*| \leq 5}} \left(\int_{H_s} |Z^{I_1^*} \partial_{\alpha} u_{\hat{j}}|^2 \cdot |Z^{I_2^*} \partial_{\beta} u_{\check{k}}|^2 dx \right)^{1/2} \\ & \quad + K \sum_{\substack{I_1^* + I_2^* = I^* \\ |I_2^*| \leq 3, |I_1^*| \leq 5}} \left(\int_{H_s} |Z^{I_1^*} \partial_{\alpha} u_{\hat{j}}|^2 \cdot |Z^{I_2^*} \partial_{\beta} v_{\check{k}}|^2 dx \right)^{1/2} \\ & \quad + K \sum_{\substack{I_1^* + I_2^* = I^* \\ |I_2^*| \geq 6}} \left(\int_{H_s} |Z^{I_1^*} \partial_{\alpha} u_{\hat{j}}|^2 \cdot |Z^{I_2^*} \partial_{\beta} u_{\check{k}}|^2 dx \right)^{1/2} \\ & \quad + K \sum_{\substack{I_1^* + I_2^* = I^* \\ |I_1^*| \geq 6}} \left(\int_{H_s} |Z^{I_1^*} \partial_{\alpha} u_{\hat{j}}|^2 \cdot |Z^{I_2^*} \partial_{\beta} v_{\check{k}}|^2 dx \right)^{1/2} \\ & \leq C(n)(C_1 \varepsilon)^2 s^{-1} + C(n)C_1 \varepsilon K s^{-1} \sum_{\substack{i \\ 6 \leq |I'| \leq 7}} E_m(s, Z^{I'} u_i)^{1/2}. \end{aligned}$$

For the second term:

$$\begin{aligned}
& \left(\int_{H_s} |Z^{I^*} (P_i^{\alpha\beta\check{j}\check{k}} \partial_\alpha u_{\check{j}} \partial_\beta v_{\check{k}})|^2 dx \right)^{1/2} \\
& \leq K \sum_{\substack{I_1^* + I_2^* = I^* \\ |I_1^*| \leq 3, |I_2^*| \leq 6}} \left(\int_{H_s} |Z^{I_1^*} \partial_\alpha u_{\check{j}}|^2 \cdot |Z^{I_2^*} \partial_\beta v_{\check{k}}|^2 dx \right)^{1/2} + K \left(\int_{H_s} |\partial_\alpha u_{\check{j}}|^2 \cdot |Z^{I^*} \partial_\beta v_{\check{k}}|^2 dx \right)^{1/2} \\
& \quad + K \sum_{\substack{|I_2^*| \leq 3 \\ I_1^* + I_2^* = I^*}} \left(\int_{H_s} |Z^{I_1^*} \partial_\alpha u_{\check{j}}|^2 \cdot |Z^{I_2^*} \partial_\beta v_{\check{k}}|^2 dx \right)^{1/2} \\
& \leq KC(n)(C_1\varepsilon)^2 s^{-3/2+\delta} + KC(n)C_1\varepsilon s^{-1} \sum_{\check{k}} E_m(s, Z^{I^*} v_{\check{k}})^{1/2} + KC(n)(C_1\varepsilon)^2 s^{-3/2+2\delta}.
\end{aligned}$$

The estimate on the third term is the same. For the forth term,

$$\begin{aligned}
& \left(\int_{H_s} |Z^{I^*} (P_i^{\alpha\beta\check{j}\check{k}} \partial_\alpha v_{\check{j}} \partial_\beta v_{\check{k}})|^2 dx \right)^{1/2} \\
& \leq K \sum_{\substack{|I_1^*| \leq 3 \\ I_1^* + I_2^* = I^*}} \left(\int_{H_s} |Z^{I_1^*} \partial_\alpha v_{\check{j}}|^2 \cdot |Z^{I_2^*} \partial_\beta v_{\check{k}}|^2 dx \right)^{1/2} + \sum_{\substack{|I_2^*| \leq 3 \\ I_1^* + I_2^* = I^*}} \left(\int_{H_s} |Z^{I_1^*} \partial_\alpha v_{\check{j}}|^2 \cdot |Z^{I_2^*} \partial_\beta v_{\check{k}}|^2 dx \right)^{1/2} \\
& \leq KC(n)(C_1\varepsilon)^{-3/2+2\delta}.
\end{aligned}$$

The estimates on terms about $Q_i^{\alpha\check{j}\check{k}}$ and $R_i^{\check{j}\check{k}}$ are similar. One omits the details.

Now one will prove (3.24). In general one has the following decomposition:

$$[G_i^{j\alpha\beta} \partial_{\alpha\beta}, Z^{I^*}] = \sum_{\substack{|I_1^*| \geq 1 \\ I_1^* + I_2^* = I^*}} Z^{I_1^*} G_i^{j\alpha\beta} Z^{I_2^*} \partial_{\alpha\beta} w_j + G_i^{j\alpha\beta} [Z^{I^*}, \partial_{\alpha\beta}] w_j.$$

The estimate on the second term is simple. One notices that, by (2.10), $[Z^{I^*}, \partial_{\alpha\beta}] w_j$ is finite linear combination of $\partial^J w_j$ with $|J| \leq 7$. So one has

$$\begin{aligned}
& \left(\int_{H_s} |G_i^{j\alpha\beta} [Z^{I^*}, \partial_{\alpha\beta}] w_j|^2 dx \right)^{1/2} \\
& \leq \sum_{|J| \leq 6} \sum_{\hat{i}} \left(\int_{H_s} |(t/s) G_i^{\hat{j}\alpha\beta}|^2 \cdot |(s/t) Z^J u_{\hat{i}}|^2 dx \right)^{1/2} + \sum_{|J| \leq 6} \sum_{\check{j}} \left(\int_{H_s} |G_i^{\check{j}\alpha\beta}|^2 \cdot |Z^J v_{\check{j}}|^2 dx \right)^{1/2} \\
& \quad + \sum_{|J| \geq 6} \sum_{\hat{i}} \left(\int_{H_s} |(t/s) G_i^{\hat{j}\alpha\beta}|^2 \cdot |(s/t) Z^J u_{\hat{i}}|^2 dx \right)^{1/2} \\
& \quad + \sum_{|J| \geq 7} \sum_{\check{j}} \left(\int_{H_s} |(t/s) G_i^{\check{j}\alpha\beta}|^2 \cdot |(s/t) Z^J v_{\check{j}}|^2 dx \right)^{1/2}.
\end{aligned}$$

Then by (3.29), one gets

$$\begin{aligned}
& \left(\int_{H_s} |G_i^{j\alpha\beta} [Z^{I^*}, \partial_{\alpha\beta}] w_j|^2 dx \right)^{1/2} \\
& \leq C(n) C_1 \varepsilon \sum_{|J| \leq 5} \sum_{\hat{i}} \left(\int_{H_s} |(t/s)(t^{-1/2}s^{-1} + t^{-3/2}s^\delta)|^2 \cdot |(s/t)Z^J u_{\hat{i}}|^2 dx \right)^{1/2} \\
& \quad + C(n) C_1 \varepsilon \sum_{|J| \leq 6} \sum_{\check{j}} \left(\int_{H_s} |t^{-1/2}s^{-1} + t^{-3/2}s^\delta|^2 \cdot |Z^J v_{\check{j}}|^2 dx \right)^{1/2} \\
& \quad + \sum_{|J| \geq 6} \sum_{\hat{i}} \left(\int_{H_s} |(t/s)(t^{-1/2}s^{-1} + t^{-3/2}s^\delta)|^2 \cdot |(s/t)Z^J u_{\hat{i}}|^2 dx \right)^{1/2} \\
& \quad + \sum_{|J| = 7} \sum_{\check{j}} \left(\int_{H_s} |(t/s)(t^{-1/2}s^{-1} + t^{-3/2}s^\delta)|^2 \cdot |(s/t)Z^J v_{\check{j}}|^2 dx \right)^{1/2} \\
& \leq C(n)(C_1 \varepsilon)^2 K s^{-1} + C(n)(C_1 \varepsilon) K s^{-3/2+\delta} + C(n)(C_1 \varepsilon) K s^{-3/2+2\delta} \\
& \quad + C(n) C_1 \varepsilon K s^{-1} \sum_{|J| \geq 6} \sum_{\hat{i}} E_m(s, Z^J u_{\hat{i}})^{1/2} + C(n) C_1 \varepsilon K s^{-1} \sum_{|J| = 7} \sum_{\check{j}} E_m(s, Z^J v_{\check{j}})^{1/2} \\
& \leq C(n)(C_1 \varepsilon) K s^{-3/2+2\delta} + C(n) C_1 \varepsilon K s^{-1} \left(\sum_{|J| \geq 6} \sum_{\hat{i}} E_m(s, Z^J u_{\hat{i}})^{1/2} + \sum_{|J| = 7} \sum_{\check{j}} E_m(s, Z^J v_{\check{j}})^{1/2} \right)
\end{aligned}$$

The estimate on the terms about

$$\sum_{\substack{|I_1^*| \geq 1 \\ I_1^* + I_2^* = I^*}} Z^{I_1^*} G_i^{j\alpha\beta} Z^{I_2^*} \partial_{\alpha\beta} w_j$$

is similar to that of (3.23). With the aid of (3.29):

$$\begin{aligned}
& \sum_{\substack{|I_1^*| \geq 1 \\ I_1^* + I_2^* = I^*}} \left(\int_{H_s} |Z^{I_1^*} G_i^{j\alpha\beta} Z^{I_2^*} \partial_{\alpha\beta} w_j|^2 dx \right)^{1/2} \\
& \leq \sum_{\substack{|I_2^*| = 6 \\ I_1^* + I_2^* = I^*}} \left(\int_{H_s} |Z^{I_1^*} G_i^{j\alpha\beta} Z^{I_2^*} \partial_{\alpha\beta} w_j|^2 dx \right)^{1/2} + \sum_{\substack{|I_2^*| = 5 \\ I_1^* + I_2^* = I^*}} \left(\int_{H_s} |Z^{I_1^*} G_i^{\hat{j}\alpha\beta} Z^{I_2^*} \partial_{\alpha\beta} u_{\hat{j}}|^2 dx \right)^{1/2} \\
& \quad + \sum_{\substack{2 \leq |I_1^*| \leq 3 \\ I_1^* + I_2^* = I^*}} \left(\int_{H_s} |Z^{I_1^*} G_i^{\check{j}\alpha\beta} Z^{I_2^*} \partial_{\alpha\beta} v_{\check{j}}|^2 dx \right)^{1/2} + \sum_{\substack{|I_1^*| = 3 \\ I_1^* + I_2^* = I^*}} \left(\int_{H_s} |Z^{I_1^*} G_i^{\hat{j}\alpha\beta} Z^{I_2^*} \partial_{\alpha\beta} u_{\hat{j}}|^2 dx \right)^{1/2} \\
& \quad + \sum_{\substack{1 \leq |I_2^*| \leq 3 \\ I_1^* + I_2^* = I^*}} \left(\int_{H_s} |Z^{I_1^*} G_i^{j\alpha\beta} Z^{I_2^*} \partial_{\alpha\beta} w_j|^2 dx \right)^{1/2} + \left(\int_{H_s} |Z^{I^*} G_i^{j\alpha\beta} \partial_{\alpha\beta} w_j|^2 dx \right)^{1/2}
\end{aligned}$$

Now take into account (3.12), (3.14) and (3.28),

$$\begin{aligned}
& \sum_{\substack{|I_1^*| \geq 1 \\ I_1^* + I_2^* = I^*}} \left(\int_{H_s} |Z^{I_1^*} G_i^{j\alpha\beta} Z^{I_2^*} \partial_{\alpha\beta} w_j|^2 dx \right)^{1/2} \\
& \leq KC(n) C_1 \varepsilon s^{-1} \sum_{|I|=7} E_m(s, Z^I w_i)^{1/2} + \left(KC(n) C_1 \varepsilon s^{-1} \sum_{\substack{i \\ |I| \geq 6}} E_m(s, Z^I w_i)^{1/2} + KC(n) (C_1 \varepsilon)^2 s^{-1} \right) \\
& \quad + KC(n) (C_1 \varepsilon)^2 s^{-3/2+2\delta} + KC(n) (C_1 \varepsilon)^2 s^{-1} \\
& \quad + KC(n) (C_1 \varepsilon)^2 s^{-1} + KC(n) C_1 \varepsilon s^{-1} \sum_i \sum_{|I| \geq 6} E_m(s, Z^I w_i).
\end{aligned}$$

So finally one concludes by (3.24). \square

Proof of lemma 3.8. One will firstly prove the following estimate:

$$\int_{H_s} (s/t) \partial_t Z^{I^*} w_i \partial_\beta Z^{I^*} w_j \partial_\alpha G_i^{j\alpha\beta} dx \leq C(n) C_1 \varepsilon s^{-1} \sum_k E_m(s, Z^{I^*} w_k).$$

By (3.29),

$$|\partial_\alpha G_i^{j\alpha\beta}| \leq C(n) C_1 \varepsilon K (t^{-1/2} s^{-1} + t^{-3/2} s^\delta).$$

Substitute this into the expression, one gets:

$$\begin{aligned}
& \int_{H_s} (s/t) \partial_t Z^{I^*} w_i \partial_\beta Z^{I^*} w_j \partial_\alpha G_i^{j\alpha\beta} dx \\
& \leq C(n) C_1 \varepsilon K \int_{H_s} (t^{-1/2} s^{-1} + t^{-3/2} s^\delta) (t/s) \cdot |(s/t) \partial_t Z^{I^*} w_i| \cdot |(s/t) Z^{I^*} \partial_\beta w_j| dx \\
& \leq C(n) C_1 \varepsilon K s^{-1} \sum_k E_m(s, Z^{I^*} w_k).
\end{aligned}$$

\square

A Local existence for small initial data

One will establish the following local-in-time existence result for small initial data. The interest is to get an a priori estimate on the life spin time. Consider the Cauchy problem in \mathbb{R}^{n+1} :

$$\begin{aligned}
(A.1) \quad & \begin{cases} g_i^{\alpha\beta}(w, \partial w) \partial_{\alpha\beta} w_i + D_i^2 w_i = F_i(w, \partial w), \\ w_i(B+1, x) = \varepsilon' w_{i0}, \quad \partial_t w_i(B+1, x) = \varepsilon' w_{i1}. \end{cases}
\end{aligned}$$

Here

$$\begin{aligned}
g_i(w, \partial w) &= m^{\alpha\beta} A_i^{\alpha\beta\gamma j} \partial_\gamma w_j + B^{\alpha\beta j} w_j + O(|w|^2 + |\partial w|^2), \\
F_i(w, \partial w) &= P_i^{\alpha\beta j k} \partial_\alpha w_j \partial_\beta w_k + Q_i^{\alpha j k} \partial_\alpha w_j w_k + R_i^{j k} w_j w_k + O(|w|^3 + |\partial w|^3).
\end{aligned}$$

These $A_i^{\alpha\beta\gamma j}, B^{\alpha\beta j}, P_i^{\alpha\beta j k}, Q_i^{\alpha j k}, R_i^{j k}$ are constants. $(w_{i0}, w_{i1}) \in H^{s+1} \times H^s$ functions and supported on the disc $\{|x| \leq B\}$. In general the following local-in-time existence holds

Theorem A.1. *For any integer $s \geq 2p(n) - 1$, there exists a time interval $[0, T(\varepsilon')]$ on which the cauchy problem (A.1) has an unique solution in sense of distribution $w_i(t, x)$. Further more*

$$w_i(t, x) \in C([0, T(\varepsilon')], H^{s+1}) \cap C^1([0, T(\varepsilon')], H^s),$$

and when ε' sufficiently small,

$$T(\varepsilon') \geq C(A\varepsilon')^{-1/2}$$

where A is a constant depending only on w_{i0} and w_{i1} . Let $E_g(T, w_i)$ be the hyperbolic energy defined in the section 2.2. For any $\varepsilon, C_1 > 0$, there exists an ε' such that

$$\sum_i E_g(B+1, w_i) \leq C_1 \varepsilon.$$

Proof. The proof is just a classical iteration procedure. The high-order terms will be omitted. One will not give the details but the key steps. One defines the standard energy associated to a curved metric g

$$E_g^*(s, w_i) := \int_{\mathbb{R}^n} (g^{00}(\partial_t u)^2 - g^{ij} \partial_i u \partial_j u) dx.$$

One takes the following iteration procedure:

$$(A.2) \quad \begin{cases} g_i^{\alpha\beta}(w^k, \partial w^k) \partial_{\alpha\beta} w_i^{k+1} = F(w^k, \partial w^k), \\ w_i(0, x) = \varepsilon' w_{i0}, \quad \partial_t w_i(0, x) = \varepsilon' w_{i0}, \end{cases}$$

and take w_i^0 as the solution of the following linear Cauchy problem:

$$\begin{cases} \square w_i = 0, \\ w_i(0, x) = \varepsilon' w_{i0}, \quad \partial_t w_i(0, x) = \varepsilon' w_{i1}. \end{cases}$$

Suppose that for any $|I| \leq 2p(n) - 1$,

$$(A.3) \quad \begin{aligned} \varepsilon' A &\geq e \cdot E_g^*(B+1, \partial^I w_i^k)^{1/2}, \\ \varepsilon' A &\geq E_g^*(t, \partial^I w_i^k)^{1/2}. \end{aligned}$$

Taking the size of the support of the solution $w_i^k(t, \cdot)$ into consideration, by Sobolev's inequality, for any $|J| \leq p(n) - 1$,

$$(A.4) \quad |\partial^J w_i^k|(t, x) \leq C(t + B + 1) \varepsilon' A.$$

Now one wants to get the energy estimate on $\partial^I w_i^{k+1}$. By the same method used in [6], one gets

$$\begin{aligned} E_g^*(t, \partial^I w_i^{k+1})^{1/2} &\leq E_g(t, \partial^I w_i^{k+1}) \exp \left(CA \varepsilon' \int_{B+1}^t (\tau + B + 1) d\tau \right) \\ &\leq e^{-1} \varepsilon' A \exp \left(CA \varepsilon' \int_{B+1}^t (\tau + B + 1) d\tau \right) \end{aligned}$$

When

$$\sqrt{CA \varepsilon'} \leq (B + 1)^{-1}$$

and

$$t \leq \frac{1}{3} (CA \varepsilon')^{-1/2},$$

one gets that

$$E_g^*(t, \partial^I w_i^{k+1})^{1/2} \leq \varepsilon' A.$$

Then by an standard method presented in the proof of theorem ... of [6],

$$\lim_{k \rightarrow \infty} w_i^k = w_i$$

is the unique solution of (A.1), and $w_i \in C([0, T(\varepsilon')], H^{s+1}) \cap C^1[0, T(\varepsilon')], H^s)$. Here one can take

$$T(\varepsilon') = C(A \varepsilon)^{-1/2}$$

To estimate $E_g(B+1, Z^I w_i)$, one takes $\partial_t w_i$ as the multiplier and by the standard procedure of energy estimate,

$$\begin{aligned} E_g(B+1, Z^I w_i) - E_g^*(B+1, Z^I w_i) &= \int_{V(B)} (Z^I F_i(w, \partial w) \partial_t w_i - [Z^I, g^{\alpha\beta} \partial_{\alpha\beta}] w_i \cdot \partial_t w_i) dx \\ &\quad + \int_{V(B)} \left(\partial_\alpha g^{\alpha\beta} \partial_t w_i \partial_\beta w_i - \frac{1}{2} \partial_t g^{\alpha\beta} \partial_\alpha w_i \partial_\beta w_i \right) dx, \end{aligned}$$

where $V(B) := \{(t, x) : t \geq B+1, t^2 - |x|^2 \leq B+1\} \cap \Lambda'$. When $A\varepsilon' \leq (B+1)^{-2}$, thanks to (A.4) and (A.3), the right hand side can be controlled by $CA\varepsilon'$. Then one gets

$$E_g(B+1, Z^I w_i) \leq CA\varepsilon'.$$

□

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